1. ∞ - Categories

1.1. Simplicial Sets

$$\underset{\text{圏}}{\Delta} := \begin{cases} \text{対象} : [n] = \{0 < 1 < \cdots < n\} & \text{for } n \geq 0 \\ \text{射} : f : [n] \to [m] & \text{s.t. } i \leq j \Rightarrow f(i) \leq f(j) \end{cases}$$

Def. 1.1.1 (Simplicial Set)

$SSet := \mathrm{Fun}(\Delta^{op}, Set) = \widehat{\Delta}$ とおく.

$SSet$ の対象を simplicial set,
$SSet$ の射を simplicial set の射という.
↪ 関手としての自然変換.

Notation. X : sset

- $n \geq 0$ に対し $X_n := X([n])$
- $\alpha : [m] \to [n]$ に対し $\alpha^* := X(\alpha) : X_n \to X_m$
- $n \geq 0$ に対し n-simplices $:= X_n$ の元

Example. 1.1.2 (Constant sset)

- X : set に対し,

$$c(X) := \begin{cases} [n] \mapsto X \\ \alpha \mapsto id_X \end{cases}$$

は sset になる.
constant simplicial set on X という.

- これによって $c : Set \to SSet$ が定まり 忠実充満.

Notation 1.1.3. X : sset

$n \geq 0$ と $0 \leq i \leq n$

$$\Delta \ni [n-1] \xrightarrow[\text{iをskip}]{\delta_n^i} [n] \quad \overset{X}{\longmapsto} \quad Set \ni X_{n-1} \xleftarrow{d_n^i} X_n \text{ : face map}$$

$$[n+1] \xrightarrow[\text{iを重ねる}]{\sigma_n^i} [n] \quad \overset{X}{\longmapsto} \quad X_{n+1} \xleftarrow{s_n^i} X_n \text{ : degeneracy map}$$

Example 1.1.5.

$n \geq 0$ に対し、

$$\underset{\text{sset}}{\Delta^n} := \mathrm{Hom}_{\Delta}(-,[n])$$ を standard n-simplex という。

Remark 1.1.6. X : sset に対し、 $\mathrm{Hom}_{\mathrm{Set}^{\Delta^{op}}}(\Delta^n, X) \cong X_n$ (米田)

$$\tilde{\sigma} \mapsfrom \sigma$$

Example 1.1.7. $n \geq 0$. $0 \leq k \leq n$

$\delta_n^k \in \mathrm{Hom}_{\Delta}([n-1],[n]) = \Delta^n_{n-1}$ より、 $\tilde{\delta}_n^k : \Delta^{n-1} \to \Delta^n$

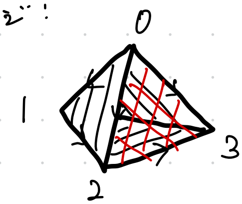
$$\underset{\text{sset}}{\partial^k \Delta^n} := \mathrm{Im}(\tilde{\delta}_n^k)$$: kth face,

$$\partial \Delta^n = \bigcup_{k=0}^{n} \partial^k \Delta^n$$: boundary as $\subseteq \Delta^n$

Example 1.1.8.

$$\underset{\text{sset}}{\Lambda^n_k} := \bigcup_{j \neq k} \partial^j \Delta^n$$: kth horn

イメージ：



1th horn

1th face

$$\begin{cases} \text{inner horn} : 0 < k < n \\ \text{outer horn} : k = 0, n \end{cases}$$ の Λ^n_k

1.2. Categories as simplicial sets.

圏 → 単体的集合

Construction 1.2.1.

$\mathcal{C}$: cat に対し

$$N(\mathcal{C}) := \mathrm{Fun}(-, \mathcal{C})$$

sset : nerve of $\mathcal{C}$

Remark 1.2.2. nerve の n-単体は 圏での $c_0 \to \cdots \to c_n$ という図式.

Exercise 1.2.3.

- $\mathcal{C} \mapsto N(\mathcal{C})$ が $N(-) : \mathrm{Cat} \to \mathrm{SSet}$: 忠実充満 を定める.
- $N(-)$ には 左随伴 $\tau : \mathrm{SSet} \to \mathrm{Cat}$ がある.

$$\begin{tikzcd} \mathrm{SSet} \arrow[dr, bend left, "\tau"] & \\ \mathbb{\Delta} \arrow[u, "Y"] \arrow[r, "F"'] & \mathrm{Cat} \arrow[ul, "N", "\perp"'] \end{tikzcd}$$

$[n] \longmapsto [n]$ as cat

- $N = \mathrm{Lan}_F Y$ である(要証明)
- $\tau := \mathrm{Lan}_Y F$ とする(Cat : 余完備より存在)

と. $\tau \dashv N$.

いわゆる普遍随伴.

Def. 1.2.4. X : sset.

(i) X の"対象" $\iff$ X の 0-単体 $\iff$ $\Delta^0 \to X$

(ii) X の"射" $\iff$ X の 1-単体 $\iff$ $\Delta^1 \to X$

(iii) f : 射, source $\iff$ $d_1^1(f) \in X_0$ $\iff$ $\Delta^0 \xrightarrow{\tilde{d}_1^1} \Delta^1 \xrightarrow{\tilde{f}} X$

target $\iff$ $d_1^0(f) \in X_0$ $\iff$ $\Delta^0 \xrightarrow{\tilde{d}_1^0} \Delta^1 \xrightarrow{\tilde{f}} X$

対象

(iv) x : 対象, id_x $\iff$ $s_0^0(x) \in X_1$ $\iff$ $\Delta^1 \xrightarrow{\tilde{s}_0^0} \Delta^0 \xrightarrow{\tilde{x}} X$

射

Notation 1.2.5.

$s = d_1^1$, $t = d_1^0$ で source, target と略記。 f : 射が $s(f) = x$, $t(f) = y$ なら $f : x \to y$ とかく.

☆ 射の合成がまだ足りない

Remark 1.2.6.

$$\begin{tikzcd}
\Delta^0 \arrow[r, "\tilde{d}_1"] \arrow[d, "\tilde{d}_1^0"'] & \Delta^1 \arrow[d] \arrow[ddr, bend left] & \\
\Delta^1 \arrow[r] \arrow[drr, bend right] & \Lambda_1^2 \arrow[dr] & \\
& & X
\end{tikzcd}$$

より.

$$\begin{aligned}
\mathrm{Hom}(\Lambda_1^2, X) &= \mathrm{Hom}(\Delta^1, X) \underset{\Delta^0}{\times} \mathrm{Hom}(\Delta^1, X) \\
&\cong X_1 \underset{X_0}{\times} X_1 \\
&= \{ (f, g) : X\text{の射}2\text{つ} \mid t(f) = s(g) \} \\
&= \{ \circ \xrightarrow{f} \circ \xrightarrow{g} \circ \text{ in } X \}
\end{aligned}$$

$\Updownarrow$

Def. 1.2.7 X : sset

(i) composable pair of morphisms $\Leftrightarrow$ $\Lambda_1^2 \to X$.

(ii) $\sigma : \Lambda_1^2 \to X$ の合成 $\Leftrightarrow$ Δ^2 への σ の延長. つまり

$$\begin{tikzcd}
\Lambda_1^2 \arrow[r, "\sigma"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, "\tilde{\sigma}"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] &
\end{tikzcd}$$

(iii) より一般に $\sigma : \Lambda_k^n \to X$ の合成 $\Leftrightarrow$ Δ^n への σ の延長.

ただし $n \geq 2$. $0 < k < n$ (inner horn)

$\therefore$ $\underbrace{\circ \xrightarrow{f} \circ \xrightarrow{g} \circ}_{\text{inner horn}}$ の合成 $\Leftrightarrow$ $\begin{tikzcd}[column sep=small] & \circ \arrow[dr, "g"] & \\ \circ \arrow[ur, "f"] \arrow[rr] & & \circ \end{tikzcd}$ への拡張

Prop. 1.2.8 (Grothendieck-Segal) X : sset. TFAE

(i) $X \cong N(\mathcal{C})$ なる圏 $\mathcal{C}$ がある.

(ii) $\forall n \geq 2$, $\forall 0 < k < n$.

$\mathrm{Hom}(\Delta^n, X) \longrightarrow \mathrm{Hom}(\Lambda_k^n, X)$ は bij. $\Leftrightarrow$

$$\begin{tikzcd}
\overset{\text{inner and}}{\Lambda_k^n} \arrow[r, "\forall \sigma"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, "\exists! \sigma'"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] &
\end{tikzcd}$$

つまり.

Nの像は inner horn の一意拡張条件で特徴づけられる.

1.3. Groupoids and Kan complexes.

Remark 1.3.1

$N|_{Grpd}$ の像は outer horn も含めた一意拡張条件で特徴づけられる.

$\because$

$$\begin{tikzcd} & \circ & \\ \circ \arrow[ur, "f"] \arrow[rr, "id"'] & & \circ \end{tikzcd} \text{ outer horn} \Rightarrow \begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr] & & \circ \end{tikzcd} \ \exists! \text{ retract}$$

$$\text{outer horn } \begin{tikzcd} & \circ \arrow[dr, "f"] & \\ \circ \arrow[rr, "id"'] & & \circ \end{tikzcd} \Rightarrow \exists! \text{ section } \begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr] & & \circ \end{tikzcd}$$

$\Longleftrightarrow$ f: 可逆

Def. 1.3.2

X: sset

X が Kan complex $\overset{def}{\Longleftrightarrow}$ $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$ が surj. ($n \geq 1$, $0 \leq k \leq n$)

$\Updownarrow$

inner も outer も

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall\sigma"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & X \\ \Delta^n \arrow[ur, "\exists\sigma'"'] & \end{tikzcd}$$

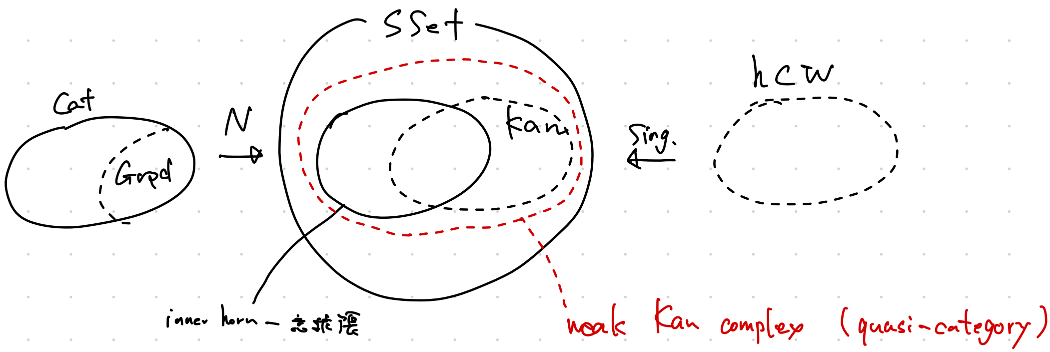
$\exists\sigma'$: 一意とは限らない.

すると $N(\mathcal{C})$ が Kan $\Longleftrightarrow$ $\mathcal{C}$: Grpd.

Theorem. 1.3.3 (Milnor)

h(CW複体の圏) $\xrightarrow{\text{Sing.}}$ (Kan complexの圏) は圏同値.

(ここで $\mathrm{Sing}(X)_n = \mathrm{Hom}_{Top}(\Delta^n_{top}, X)$
h は weak homotopy equivalence での局所化



1.4 ∞-Categories as weak Kan complexes.

Def. 1.4.1 (Boardman-Vogt). X : SSet

X が weak-Kan complex ⇔ (def) $n \geq 2,\ 0<k<n$
quasi-category

$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$ が surj

⇕

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall\sigma"] \arrow[d] & X \\ \Delta^n \arrow[ur, "\exists\sigma'"'] \arrow[u, phantom, ""] & \end{tikzcd}$$

inner Λ^n_k, ↻, $\exists\sigma'$: 一意とは限らない

Construction 1.4.2 (homotopy category)

∘ X : wKan, $f, g : x \to y$ in X

$\sigma : \Delta^2 \to X$: X の 2-単体 が

$$\begin{cases} d_2^0(\sigma) = \mathrm{id}_y \\ d_2^1(\sigma) = g \\ d_2^2(\sigma) = f \end{cases}$$

なら. これを f と g の homotopy といい. $\sigma : f \sim g$ とかく.

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] \arrow[d, phantom, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & {} & y \end{tikzcd}$$

∘ $x \to y$ なる射に同値関係が定まる

∵ 推移律 :

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr] \arrow[ddr, "h"'] & \sigma & y \arrow[dl, "\mathrm{id}_y"'] \arrow[ddl, "\mathrm{id}_y"] \\ \sigma' & y \arrow[d, "\mathrm{id}_y"] & * \\ & y & \end{tikzcd}
\overset{合成}{\Rightarrow}
\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "h"'] & \sigma,\ \sigma'* & y \arrow[dl, "\mathrm{id}_y"] \\ & y & \end{tikzcd}$$

∘ $\underset{\text{Cat}}{h(X)} :=$ $\begin{cases} \text{対象} : X \text{の"対象"} (= 0\text{-単体}) \\ \text{射} : X \text{の"射"} (= 1\text{-単体}) \text{の homotopy 類.} \end{cases}$

∵ ∘ $f : x \to y$, $g : y \to z$. σ_1, σ_2 が f と g の合成なら.

$$\begin{tikzcd} x \arrow[rr, bend left] \arrow[dr, "f"] \arrow[ddr, bend right] & \sigma_1 & z \arrow[ddl, bend left, "\mathrm{id}_z"] \\ \sigma_2 & y \arrow[ur, "g"'] \arrow[d, "g"] & * \\ & z & \end{tikzcd}
\Rightarrow
\begin{tikzcd} x \arrow[rr, "g\circ f"] \arrow[dr, "g\circ f_2"'] & \text{homotopy} & z \arrow[dl, "\mathrm{id}_z"] \\ & z & \end{tikzcd}$$

∘ $g \sim h$ なら $g \circ f \sim g \circ h$ なども同様.

h : Cat → SSet が定まり. $h \dashv N(-)$.

Def. 1.4.3. X : wKan

$f : x \to y$ が iso $\overset{\text{def}}{\iff}$ f は可逆.

$\iff$ $\exists g : y \to x$. $f \circ g \sim \mathrm{id}_y$, $g \circ f \sim \mathrm{id}_x$

Remark.
- $f : x \simeq y$ は $\begin{tikzcd} \Delta^0 \arrow[r, bend left=50, "\tilde{x}"{name=U}] \arrow[r, bend right=50, "\tilde{y}"'{name=D}] & X \arrow[from=U, to=D, phantom, "\simeq\Downarrow\tilde{f}"] \end{tikzcd}$ とみなせる.
- f が iso $\iff$ $h(X)$ で iso

Def X : wKan

X は ∞-groupoid $\iff$ X の全ての射は iso.

Theorem 1.4.4 (Joyal). X : wKan.

X は ∞-groupoid $\iff$ X は Kan

Remark 1.4.5.

(i) X, Y : wKan.

$\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathrm{Set}^{\Delta^{op}}}(\Delta^{\bullet} \times X, Y)$ は $\mathrm{Fun}(X, Y)$ とみれる.

(ii) X : wKan. x, y : X の対象

$\mathrm{Maps}_X(x, y)$ は $\mathrm{Hom}(x, y)$ とみれる. ので そう書く.

$$\left(\begin{tikzcd} \mathrm{Maps}_X(x, y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d, "(s,t)"] \\ \Delta^0 \arrow[r, "(x,y)"] & X \times X \end{tikzcd}\right)$$

Def. 1.4.6.

weak Kan complex のことを, ∞-category という

1.5. The ∞-category of ∞-groupoids.

ふつうの nerve :

$$N(\mathcal{C})_n := \left\{ \begin{array}{ll} \circ\ C_i \in \mathcal{C} & (0 \le i \le n) \\ \circ\ f_{ij} : C_i \to C_j & (0 \le i < j \le n) \text{ s.t.} \end{array} \right.$$

$$\begin{tikzcd} C_i \arrow[r, "f_{ij}"] \arrow[rr, bend right, "f_{ik}"'] & C_j \arrow[r, "f_{jk}"] \arrow[d, phantom, "||"] & C_k \\ & {} & \end{tikzcd}$$

Construction 1.5.1. $\mathcal{C}$: 2-cat (Cat - enriched cat ?)

$$N^D(\mathcal{C})_n := \left\{ \begin{array}{l} \circ\ C_i \in \mathcal{C} \quad (0 \le i \le n) \\ \circ\ f_{ij} : C_i \to C_j \quad (0 \le i < j \le n) \ \text{と} \end{array} \right.$$

$$\left[\begin{tikzcd} C_i \arrow[r, "f_{ij}"] \arrow[rr, bend right, "f_{ik}"'] & C_j \arrow[r, "f_{jk}"] \arrow[d, phantom, "\Downarrow \mu_{i,j,k}"] & C_k \\ & {} & \end{tikzcd} \right]$$

s.t.

$$\begin{tikzcd} C_j \arrow[r] \arrow[dr, phantom, "\Downarrow"] & C_k \arrow[d] \\ C_i \arrow[u] \arrow[ur] \arrow[r] \arrow[ur, phantom, "\Downarrow" near end] & C_\ell \end{tikzcd} = \begin{tikzcd} C_j \arrow[r] \arrow[dr] & C_k \arrow[d] \arrow[dl, phantom, "\Downarrow" near start] \\ C_i \arrow[u] \arrow[r] \arrow[ur, phantom, "\Downarrow" near start] & C_\ell \end{tikzcd}$$

↑ sset

Duskin nerve という.

Theorem 1.5.2 (Duskin) $\mathcal{C}$: 2cat. TFAE

(i) $\mathcal{C}$ は (2,1)-圏. つまり 全ての 2-射 が 可逆.

(ii) $N^D(\mathcal{C})$ が wKan.

Example 1.5.4

○ Grpd : groupoid の圏は 自然に 2-圏 となる.

Construction 1.5.5.

$$\underset{\mathrm{sSet}}{\mathrm{Kan}_n} = \left\{ \begin{array}{l} \circ\ K_i : \text{Kan complex}\ (0 \le i \le n), \\ \circ\ f_{ij} : K_i \to K_j\ (0 \le i < j \le n),\ \circ\ \text{coherent homotopy} \text{ の組} \end{array} \right\}$$

e.g. ∘ $\mathrm{Kan}_2 = \left\{ \begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\sigma"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd} \ \middle|\ \begin{array}{l} \sigma : \Delta^1 \times X \longrightarrow Z \\ \text{s.t.} \ \cdot\ d_1^0(\sigma) = g \circ f \\ \quad\ \ \cdot\ d_1^1(\sigma) = h \end{array} \right\}$

$\sigma \in \underline{\mathrm{Hom}}(X, Z)_1$ に注意.

∘ $n \ge 3$ では "homotopies between homotopies" が必要.

$$\mathrm{Kan}_3 = \left\{ \begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "p"] \arrow[rrr, bend left=60, "r"] \arrow[rrr, bend right=60, "r"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "q"'] & Z \arrow[r, "h"] & W \end{tikzcd} \quad + \quad \alpha : \text{“}\tau_1 \circ \sigma_1 \sim \tau_2 \circ \sigma_2\text{''} \ ? \right\}$$

Q. α は誰？ (以下 Remark 1.5.6. の解読)

準備 : ∘ sSet-enriched category を simplicial category という.
それらがなす圏を Cat_Δ で表す.

J : poset

∘ $\underset{\mathrm{Cat}_\Delta}{\mathrm{Path}(J)}$:
- ∘ 対象は J の対象
- ∘ hom-set は, $i, j \in J$ に対し
  $\mathrm{Hom}(i, j) := N(P_{ij}) \in \mathrm{SSet}$ where $\underset{\mathrm{Poset}}{P_{ij}} = \{ I \subseteq J \mid i, j \in I \text{ かつ } \forall k \in I,\ i \le k \le j \}$ + 逆包含順序
- ∘ 合成は poset $\subseteq J$ の合併で誘導.

(cf. Kerodon 2.4.3 https://kerodon.net/tag/00KM)

◦ Path[3] の展開。 対象は 0, 1, 2, 3.　　$i > j$ なら $\mathrm{Hom}(i,j) = \emptyset$.
([3]でなくても)

$P_{01} = \{\{0,1\}\}$, $P_{12} = \{\{1,2\}\}$, $P_{23} = \{\{2,3\}\}$

$P_{02} = \{\{0,1,2\} \to \{0,1\}\}$, $P_{13} = \{\{1,2,3\} \to \{1,3\}\}$

$$P_{03} = \left\{ \begin{tikzcd}[row sep=small] & \{0,1,3\} \arrow[dr] & \\ \{0,1,2,3\} \arrow[ur] \arrow[dr] & & \{0,3\} \\ & \{0,2,3\} \arrow[ur] & \end{tikzcd} \right\}$$

の各 Nerve が $\mathrm{Hom}_{\mathrm{Path}[3]}(i,j)$ となる。

Definition 2.4.3.5 (The Homotopy Coherent Nerve). Let $\mathcal{C}_\bullet$ be a simplicial category. We let $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ denote the simplicial set given by the construction

$$([n] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathrm{Path}[n]_\bullet, \mathcal{C}_\bullet) = \{\text{Simplicial functors } \mathrm{Path}[n]_\bullet \to \mathcal{C}_\bullet\}.$$

We will refer to $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ as the *homotopy coherent nerve* of the simplicial category $\mathcal{C}_\bullet$.

$\mathcal{C}$ として Kan complex のなす圏をとった様子。
($\underline{\mathrm{Hom}}(X,Y)$ を hom-set とすることで sSet-enriched になっている)

$\mathrm{Kan}_3 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathrm{Path}[3], \mathrm{Kan})$

$=$ ◦ 対象レベルで $\underset{0 \le i \le 3}{i} \mapsto K_i$: Kan complex のデータの指定。

$+$ ◦ $i \le j \in [3]$ に対し

$\underset{= N(P_{ij})}{\mathrm{Hom}_{\mathrm{Path}[3]}(i,j)} \to \underset{= \underline{\mathrm{Hom}}(K_i, K_j)}{\mathrm{Hom}_{\mathrm{Kan}}(K_i, K_j)}$: sSet の間の射。

0-単体.　$\mathrm{Fun}([0], P_{ij}) \to \mathrm{Hom}_{\Delta}(\Delta^0 \times K_i, K_j)$　$\because \Delta^0_n = \mathrm{Hom}([n],[0])$
$\cong P_{ij} \to \cong \mathrm{Hom}_{\Delta}(K_i, K_j)$　　$\cong 1$.

$$\left[\begin{array}{ccc} P_{01}, & P_{12}, & P_{23} \\ \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} \\ \{0,1\}, & \{1,2\}, & \{2,3\} \\ \{0,2\}, & \{1,3\}, & \{0,3\} \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ P_{02}, & P_{13}, & P_{03} \end{array}\right. \mapsto \begin{tikzcd} K_0 \arrow[r, "f_{01}"] \arrow[rr, bend right, "f_{02}"'] \arrow[rrr, bend right=60, "f_{03}"'] & K_1 \arrow[r, "f_{12}"] \arrow[rr, bend right, "f_{13}"'] & K_2 \arrow[r, "f_{23}"] & K_3 \end{tikzcd}$$

のデータの指定。

$$
\begin{array}{ccccc}
P_{02} & , & P_{13} & & \\
\ni & & \ni & & \\
\{0,1,2\} & , & \{1,2,3\} & \longmapsto & f_{12}\circ f_{01} \ , \ f_{23}\circ f_{12} \\
\| & & \| & & \\
\{0,1\}\text{と}\{1,2\} & & \{1,2\}\text{と}\{2,3\} & & \\
\text{の合併} & & \text{の合併} & &
\end{array}
$$

自動的に定まる.

$$
\{0,1,2,3\},\ \underset{\substack{\ni \\ P_{03}}}{\{0,1,3\}},\ \{0,2,3\} \longmapsto f_{23}\circ f_{12}\circ f_{01},\ f_{13}\circ f_{01},\ f_{23}\circ f_{02}
$$

$$
\begin{array}{lccc}
& N(P_{ij})_1 & & \underline{\mathrm{Hom}}(K_i, K_j)_1 \\
1\text{-単体} & \mathrm{Fun}([1], P_{ij}) & \to & \mathrm{Hom}_{\hat{\Delta}}(\Delta^1\times K_i, K_j) \\
& \cong \{(I\to J) \mid I, J \in P_{ij}\} & \to & \cong \quad ?
\end{array}
$$

・$(I\to I)$ の行き先は

$$
\begin{array}{ccccc}
N(P_{ij})_1 \ni & (I\to I) & \longmapsto & \text{よってここも決まる.} & \\
\uparrow s_0^0 & \uparrow & \circlearrowleft & \uparrow & \text{より自動的に定まる.} \\
N(P_{ij})_0 \ni & I & \longmapsto & \text{さっき定めた} &
\end{array}
$$

以下 $I\neq J$ なる $I\to J$.

・

$$
\begin{array}{ccc}
P_{02} \ni \{0,1,2\}\to\{0,2\} & \longmapsto & \sigma_1 : \Delta^1\times K_0 \to K_2 \\
P_{13} \ni \{1,2,3\}\to\{1,3\} & & \sigma_2 : \Delta^1\times K_1 \to K_3
\end{array}
$$

つまり

$$
\begin{tikzcd}
K_0 \arrow[r] \arrow[rr, bend right, ""{name=A}] & K_1 \arrow[r] \arrow[rr, bend right, ""{name=B}] & K_2 \arrow[r] & K_3
\arrow[from=1-2, to=A, phantom, "\Downarrow\sigma_1"]
\arrow[from=1-3, to=B, phantom, "\sigma_2\Downarrow"]
\end{tikzcd}
$$

のデータの指定.

d_i^j についての可換性条件から $d_i^j(\sigma_1)$, $d_i^j(\sigma_2)$ の条件は定まる.

・P_{03} は 5つあるが、

・$\{0,1,2,3\} \to \{0,1,3\}$ $\mapsto$
$$\begin{tikzcd} K_0 \arrow[r, shift left, "f_{01}"] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow"] & K_1 \arrow[r] \arrow[rr, bend right, "\sigma_1\Downarrow"'] & K_2 \arrow[r] & K_3 \end{tikzcd}$$

$\quad = \{0,1\} \to \{0,1\}$ と
$\quad\ \ \{1,2,3\} \to \{1,3\}$ の合成

$\sigma_1 * f_{01}$ とかく。

自動的に定まる。

・$\{0,1,2,3\} \to \{0,2,3\}$ $\mapsto$
$$\begin{tikzcd} K_0 \arrow[r] \arrow[rr, bend right, "\Downarrow\sigma_2"'] & K_1 \arrow[r] & K_2 \arrow[r, shift left, "f_{23}"] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow"] & K_3 \end{tikzcd}$$

$\quad = \{0,1,2\} \to \{0,1,2\}$ と
$\quad\ \ \{2,3\} \to \{2,3\}$ の合成

$f_{23} * \sigma_2$ とかく。

・$\{0,1,3\} \to \{0,3\}$ $\mapsto$
$$\begin{tikzcd} K_0 \arrow[r] \arrow[rrr, bend right=50, ""{name=B}] & K_1 \arrow[rr, bend right, ""{name=A}] & K_2 & K_3 \arrow[from=A, to=B, phantom, "\Downarrow\tau_1"] \end{tikzcd}$$

・$\{0,2,3\} \to \{0,3\}$ $\mapsto$
$$\begin{tikzcd} K_0 \arrow[rr, bend right] \arrow[rrr, bend right=50] & K_1 & K_2 \arrow[r] \arrow[r, phantom, ""] & K_3 \end{tikzcd}$$
$\tau_2 \Downarrow$

・$\{0,1,2,3\} \to \{0,3\}$ $\mapsto$
$$\begin{tikzcd} K_0 \arrow[r] \arrow[rrr, bend right, "\Downarrow\rho"'] & K_1 \arrow[r] & K_2 \arrow[r] & K_3 \end{tikzcd}$$

のデータの指定。

2-単体 $\mathrm{Fun}([2], P_{ij}) \to \mathrm{Hom}_S(\Delta^2 \times K_i, K_j)$
$= \{(I \to J \to K) \mid I, J, K \in P_{ij}\}$

$I = J$ や $J = K$ の場合は上と同じく
退化によって下のデータによって定まる。

∘ P_{03} において 2つある.

$$\begin{array}{lcl}
\cdot\ \{0,1,2,3\}\to\{0,1,3\}\to\{0,3\} & \mapsto & \alpha_1 : \Delta^2\times K_0 \to K_3 \\
 & \text{s.t.} & \\
d_2^0 :\ \{0,1,3\}\to\{0,3\} & \mapsto & \tau_1 \\
d_2^1 :\ \{0,1,2,3\}\to\{0,3\} & \mapsto & \rho \\
d_2^2 :\ \{0,1,2,3\}\to\{0,1,3\} & \mapsto & \sigma_1 * f_{01}
\end{array}$$

$$\rightsquigarrow\quad
\begin{tikzcd}
 & f_{13}\circ f_{01} \arrow[dr, "\tau_1"] & \\
f_{23}\circ f_{12}\circ f_{01} \arrow[ur, "\sigma_1 * f_{01}"] \arrow[rr, "\rho"'] \arrow[rr, phantom, shift left=3, "\alpha_1"] & & f_{03}
\end{tikzcd}$$

つまり

$$\begin{tikzcd}
K_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[rrr, bend right, "\tau_1"'] & K_1 \arrow[r] \arrow[rr, bend right, "\sigma_1"'] & K_2 \arrow[r] & K_3
\end{tikzcd}$$

$$\alpha_1\ \wr$$

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[rrr, bend right, "\rho"'] & K_1 \arrow[r] & K_2 \arrow[r] & K_3
\end{tikzcd}$$

のデータの指定.

$$\begin{array}{lcl}
\cdot\ \{0,1,2,3\}\to\{0,2,3\}\to\{0,3\} & \mapsto & \alpha_2 : \Delta^2\times K_0 \to K_3 \\
 & \text{s.t.} & \\
d_2^0 :\ \{0,2,3\}\to\{0,3\} & \mapsto & \tau_2 \\
d_2^1 :\ \{0,1,2,3\}\to\{0,3\} & \mapsto & \rho \\
d_2^2 :\ \{0,1,2,3\}\to\{0,2,3\} & \mapsto & f_{23} * \sigma_2
\end{array}$$

$$\rightsquigarrow\quad
\begin{tikzcd}
 & f_{02}\circ f_{23} \arrow[dr, "\tau_2"] & \\
f_{23}\circ f_{12}\circ f_{01} \arrow[ur, "f_{23} * \sigma_2"] \arrow[rr, "\rho"'] \arrow[rr, phantom, shift left=3, "\alpha_2"] & & f_{03}
\end{tikzcd}$$

つまり

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[rr, bend right, "\sigma_2"'] \arrow[rrr, bend right=50, "\tau_2"'] & K_1 \arrow[r] & K_2 \arrow[r, shift left] \arrow[r, shift right] & K_3
\end{tikzcd}$$

$$\alpha_2\ \wr$$

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[rrr, bend right, "\rho"'] & K_1 \arrow[r] & K_2 \arrow[r] & K_3
\end{tikzcd}$$

のデータの指定.